4248. *Proposed by Michel Bataille.*

Let n be a positive integer and let $p(x) = 1 + p_1(x) + p_2(x) + \cdots + p_n(x)$ where the polynomials $p_k(x)$ are defined by $p_0(x) = 2$, $p_1(x) = x^2 + 2$ and the recursion

$$p_{k+1}(x) = (x^2 + 2)p_k(x) - p_{k-1}(x)$$

for $k \in \mathbb{N}$. Find all the complex roots of $p(x)$.

There were five correct solutions. Two of them followed the strategy of Solution 1, while the others worked from the solution of the recursion.

Solution 1, by Arkady Alt.

For $k \geq 0$, let $q_k(t) = p_k(t - 1/t)$, and let $q(t) = p(t - 1/t)$. Then $q_0(t) = 2$ and an induction argument reveals that

$$q_k(t) = t^{2k} + \frac{1}{t^{2k}}$$

for $k \geq 1$. Hence

$$q(t) = \sum_{k=-n}^{n} t^{2k} = t^{-2n} \sum_{k=0}^{2n} t^{2k} = \frac{t^{4n+2} - 1}{t^{2n}(t^2 - 1)}.$$

For $1 \leq k \leq 2n$, let

$$t_k = \cos\left(\frac{k\pi}{2n+1}\right) + i \sin\left(\frac{k\pi}{2n+1}\right), \quad x_k = t_k - \frac{1}{t_k} = 2i \sin\left(\frac{k\pi}{2n+1}\right).$$

Then $p(x_k) = q(t_k) = 0$.

For $k \geq 0$, the degree of $p_k(x)$ is $2k$ so that the degree of $p(x)$ is $2n$. We have identified $2n$ distinct roots, x_k, of $p(x)$. Thus

$$\left\{2i \sin\left(\frac{k\pi}{2n+1}\right) : k = 1, 2, \ldots, 2n\right\}$$

is the set of roots of $p(x)$.

Solution 2, by Ivko Dimitrić.

Solving the recursion for $p_k(x)$ yields $p_k(x) = u^k + v^k$ for $k \geq 0$ where

$$u = \frac{x^2 + 2 + x\sqrt{x^2 + 4}}{2} \qquad \text{and} \qquad v = \frac{1}{u} = \frac{x^2 + 2 - x\sqrt{x^2 + 4}}{2}.$$

Therefore,when $x \neq 0$,

$$\begin{aligned} p(x) &= 1 + u\left(\frac{u^n - 1}{u - 1}\right) + v\left(\frac{v^n - 1}{v - 1}\right) \\ &= \left[\frac{u^{n+1}}{u - 1} + \frac{v^{n+1}}{v - 1}\right] + \left[1 - \frac{u(v-1) + v(u-1)}{(u-1)(v-1)}\right] \\ &= \frac{u^{n+1}}{u - 1} + \frac{v^{n+1}}{v - 1}, \end{aligned}$$